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The forcing number of toroidal polyhexes

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The forcing number, denoted by $f(G)$, of a graph G with a perfect matching is the minimum number of independent edges that completely determine the perfect matching of G. In this paper, we consider the forcing number of a toroidal polyhex $H(p,q,t)$ with a torsion t, a cubic graph embedded on torus with every face being a hexagon. We obtain that $f(H(p,q,t)) \geqslant \min\{p,q\}$, and equality holds for $p \leqslant q$ or $p > q$ and $t \in \{0, p-q, p-q+1, \ldots, p-1\}$. In general, we show that $f(H(p,q,t))$ is equal to the side length of a maximum triangle on $H(p,q,t)$. Based on this result, we design a linear algorithm to compute the forcing number of $H(p,q,t)$.

KEY WORDS: toroidal polyhex, forcing number, kekulé structure, perfect matching

1. Introduction

The concept of forcing number of benzenoids was first proposed by Harary et al. [4]. The same idea appeared in earlier papers by Randić and Klein [15] and Klein and Randić [5] in terms of "innate degree of freedom" of a Kekulé structure. The benzenoids with forcing number 1 was investigated in [21–24]. The forcing number of Buckminsterfullerene (C_{60}) have been given by Vukičević et al. [18].

In this paper we gives a fast computation for the forcing number of a toroidal polyhex, or toroidal fullerene, a cubic bipartite graph on torus such that every face is a hexagon [9], which can be denoted by $H(p,q,t)$ for a string (p,q,t) of three integers ($p \geqslant 1, q \geqslant 1, 0 \leqslant t \leqslant p-1$). In 1997, the "Crop circles fullerenes" discovered by Liu et al. [13] has been presumably torus-shaped.

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Enumeration of ismoers [10], Kekulé structure count [6], spanning tree count [8], and chirality [11] of toroidal ployhexes have been investigated.

Our approach relys on a general research on the forcing number of a bipartite graph with a perfect matching (or Kekulé structure in chemistry). Let G be a graph with a perfect matching M. A set $S \subset M$ is called a *forcing set* of M if S is not contained in any other perfect matchings of G. The *forcing number* of M, denoted by $f(G, M)$, is the minimum size of forcing sets of M. The *forcing number* of G, denoted by $f(G)$, is the minimum value of the forcing numbers of all perfect matchings of G.

Recently, the forcing numbers of bipartite graphs [1, 2], in particular, for square grids [14], stop signs [12], torus, and hypercube [1, 7, 16], have been considered. Riddle [16] gave a lower bound of the forcing number of bipartite graphs. Adams et al. [1] proved it is NP-complete to find the smallest forcing set of a bipartite graph with maximum degree 3. Most recently, the global forcing number and anti-forcing number of benzenoids were introduced by Došlić [3], Vukičević and Sedlar [19], and Vukičević and Trinajstić [20], respectively.

By improving Riddle's method which produces a lower bound of the forcing number of bipartite graphs we determine the forcing numbers of toroidal polyhexes $H(p, q, t)$. For the degenerated cases: $H(1, q, 0)$, $H(p, 1, 0)$, and $H(p, 1, p-1)$, their forcing numbers equal one. From now on we suppose that toroidal polyhexes in question always means the non-degenerated cases. We prove that $f(H(p, q, t)) \geqslant \min\{p, q\}$, and equality holds for $p \leqslant q$ or $p > q$ and $t \in \{0, p-q, p-q+1, \ldots, p-1\}$. Generally, we prove that $f(H(p, q, t))$ is equal to the side length of a maximum triangle on $H(p, q, t)$. Finally, we design a fast algorithm of $O(n)$ times to compute the forcing number of $H(p, q, t)$, where n is the number of vertices of $H(p, q, t)$.

2. Preliminaries for toroidal polyhex

A *toroidal polyhex* $H(p, q, t)$ can be defined as: Let P be a $p \times q$ parallelogram section cut from the hexagonal lattice such that every corner lies on the center of a hexagon, two lateral sides pass through q oblique edges, top and bottom sides pass through p vertical edges; Identify two lateral sides of P to form a cylinder, and then identify top and bottom sides with a torsion t hexagons (see figure 1). The toroidal polyhex $H(p, q, t)$ is a bipartite graph since the vertices admit a proper 2-coloring: the vertices incident with a downward vertical edge and two upwardly lateral edges are colored by white, and the other vertices black.

We put $H(p, q, t)$ in an affine coordinate system XOY: take the bottom side as x-axis and a lateral side as y-axis such that x-axis and y-axis form an angle with 60°, the origin O is their intersection and P lies on non-negative region (see figure 1). We take the distance between a pair of parallel edges in

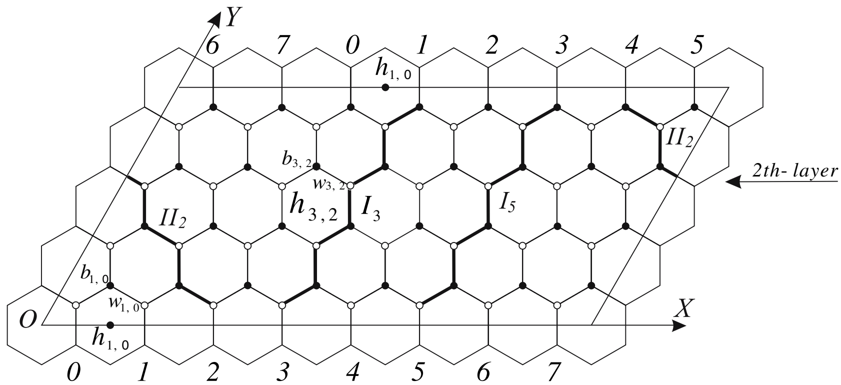

Figure 1. Toroidal polyhex $H(8, 4, 2)$, I-column (I_3 and I_5) and II-column (II_2).

a hexagon as unit length. Each hexagon is labeled by the coordinates (x, y) of its center, and denoted by (x, y) or $h_{x,y}$, where $x \in \mathbb{Z}_p := \{0, 1, \ldots, p-1\}$ and $y \in \mathbb{Z}_q := \{0, 1, \ldots, q-1\}$. In a hexagon $h_{x,y}$, label the ends of the upper one in the two edges vertical to y-axis by $b_{x,y}$ and $w_{x,y}$ with respecting to the vertex colors (see figure 1). Under this labeling, each $w_{0,y}$ is adjacent to $b_{0,y}$ and $w_{x,0}$ is adjacent to $b_{x+t+1,q-1}$ ($y \in \mathbb{Z}_q, x \in \mathbb{Z}_p$). The yth layer is defined to be the even cycle $w_{0,y}b_{1,y}w_{1,y}b_{2,y} \ldots w_{p-1,y}b_{0,y}w_{0,y}$ ($y \in \mathbb{Z}_q$).

An automorphism ϕ of a graph is a bijection from the vertex set to itself which satisfies both ϕ and the inverse ϕ^{-1} preserve the adjacency between vertices. A graph is *vertex-transitive* if there exists an automorphism between any two vertices. Let $\phi_{rl}(v_{x,y}) = v_{x-1,y}$ and $\phi_{tb}(v_{x,y}) = v_{x,y-1}$ where $v \in V(H(p, q, t))$, $x-1, x \in \mathbb{Z}_p$ and $y-1, y \in \mathbb{Z}_q$, then ϕ_{rl} and ϕ_{tb} are two automorphisms of $H(p, q, t)$ (see [17]).

The path $w_{x,0}b_{x+1,0}w_{x,1}b_{x+1,1} \ldots w_{x,q-1}b_{x+1,q-1}$ is called I_x-*column* ($0 \leqslant x \leqslant p-1$), simply denoted by I_x, and the vertices $w_{x,0}$ and $b_{x+1,q-1}$ are called the *head* and the *tail* of I_x, respectively. The I_{x_2} is called the *successor* of I_{x_1} if the tail of I_{x_1} is adjacent to the head of I_{x_2}, and thereby $w_{x_2,0}$ is adjacent to $b_{x_1+1,q-1}$, further $x_2 \equiv x_1 - t \pmod{p}$ (see figure 1, I_3 is the successor of I_5). Let $I_{x_1}, I_{x_2}, \ldots, I_{x_g}$ be g different I-columns such that $I_{x_{i+1}}$ is the successor of I_{x_i} for $i, i+1 \in \mathbb{Z}_g$, then these g I-columns form a cycle, called I-*cycle*. According to the automorphism ϕ_{rl}, every I-column lies in an I-cycle and every I-cycle contains the same number of I-columns. The direction of a I-cycle is called I^+-*direction* if it is from head to tail along every I-column of this cycle, and the another direction is called I^--*direction*.

The path $w_{x,0}b_{x,0}\ w_{x-1,1}b_{x-1,1} \ldots w_{x-q+1,q-1}b_{x-q+1,q-1}$ is called II_x-*column* ($0 \leqslant x \leqslant p-1$). Similarly, we can define the *head, tail, successor* of a II-column and II-*cycle*, II^+-*direction*, II^--*direction* as these defined for a

I-column. If II_{x_2} is the successor of II_{x_1}, then $w_{x_2,0}$ is adjacent to $b_{x_1-q+1,q-1}$, further $x_2 \equiv x_1 - (q+t) \pmod p$.

Lemma 2.1. If a I-cycle (resp. II-cycle) of $H(p,q,t)$ has g I-columns (II-columns), then:

(a) $H(p,q,t)$ has $\frac{p}{g}$ I-direction cycles where g is the smallest positive integer satisfying $gt \equiv 0 (\mathrm{mod}\ p)$ and any consecutive $\frac{p}{g} I$-columns: $I_x, \ldots, I_{x+\frac{p}{g}-1}$ lie on distinct I-cycles.

(b) $H(p,q,t)$ has $\frac{p}{g}$ II-direction cycles where g is the smallest positive integer satisfying $g(q+t) \equiv 0 (\mathrm{mod}\ p)$ and any consecutive $\frac{p}{g} II$-columns: $II_x, \ldots, II_{x+\frac{p}{g}-1}$ lie on distinct II-cycles.

Proof. (a) According to the automorphism ϕ_{rl}, it suffices to consider the I-cycle consisting of $I_0, I_{p-t}, \ldots, I_{p-(g-1)t}$. By the definition of I-cycle, the head of I_0 is adjacent to the tail of $I_{p-(g-1)t}$, then $w_{0,0}b_{p-(g-1)t+1,q-1} \in E(H(p,q,t))$, then $p-gt=0$ and $p-rt \neq 0$ for $r<g$. Hence, g is the smallest positive integer satisfying $gt \equiv 0 \pmod p$. If $t=0$, then $g=1$. Hence, $H(p,q,t)$ has p I-cycles and the assertion holds. So suppose $t \neq 0$, then $g \geqslant 2$. Since $H(p,q,t)$ has p I-columns and every I-column lies on one I-cycle, every I-cycle contains $\frac{p}{g}$ I-columns.

In the following, we will prove any two I-columns from $I_0, I_1, \ldots, I_{\frac{p}{g}-1}$ do not lie on the same I-direction cycle. Suppose to the contrary that I_0 and I_k with $0<k<\frac{p}{g}$ lies on the same I-cycle by the automorphism ϕ_{rl}. Then $p-rt \equiv k \pmod p$ with $0<r \leqslant g-1$ and further $rt \equiv k \pmod p$. Therefore, there exists $\mu \in \mathbb{Z}^+$ such that $\mu p - rt = k$, further $1 \leqslant \mu p - rt \leqslant \frac{p}{g}-1$. Hence $2 \leqslant g \leqslant \mu p g - rtg \leqslant p-g$. According $gt \equiv 0 \pmod p$, there exists $\lambda \in \mathbb{Z}^+$ such that $\lambda p = gt$. Then $2 \leqslant g \leqslant (\mu g - \lambda r)p \leqslant p-g$, which contradicts $\mu, \lambda, g, r \in \mathbb{Z}^+$. Therefore, $I_0, \ldots, I_{\frac{p}{g}-1}$ lie on the distinct I-cycles.

(b) It suffices to consider the II-cycle consisting of $II_0, II_{p-q-t}, \ldots, II_{(g-1)(p-q-t)}$. By the definition of II-cycle, then $w_{g(p-q-t),0} = w_{0,0}$ and $w_{r(p-q-t),0} \neq w_{0,0}$ for $r<g$. So g is the smallest positive integer satisfying $g(p-q-t) \equiv 0 \pmod p$. If $p-q-t \equiv 0 \pmod p$, then $g=1$. Hence, $H(p,q,t)$ has p II-cycles and the assertion holds. So suppose $p-q-t \neq 0 \pmod p$, then $g \geqslant 2$. Since $H(p,q,t)$ has p II-columns and every II-column lies on one II-cycle, then II-direction cycle contains $\frac{p}{g} II$-columns.

In the following, we will prove any two II-columns from $II_0, II_1, \ldots, II_{\frac{p}{g}-1}$ do not lie on the same II-cycle. Suppose to the contrary that II_0 and II_k with $0<k<\frac{p}{g}$ lies on the same II-cycle by the automorphism ϕ_{rl}. Then $p-r(p-q-t) \equiv k \pmod p$ with $0<r \leqslant g-1$, further $r(p-q-t) \equiv k$ (mod p). Therefore, there exists $\mu \in \mathbb{Z}^+$ such that $r(p-q-t) = k + \mu p$, further

$1 \leqslant r(p-q-t) - \mu p \leqslant \frac{p}{g} - 1$. On the other hand, since $g(p-q-t) \equiv 0 \pmod{p}$, there exists $\lambda \in \mathbb{Z}^+$ such that $g(p-q-t) = \lambda p$. Hence $2 \leqslant g \leqslant r(p-q-t)g - \mu pg = r\lambda p - \mu pg = (r\lambda - \mu g)p \leqslant p - g$ which contradicts $\lambda, \mu, r, g \in \mathbb{Z}^+$. Therefore, $II_0, \ldots, II_{\frac{p}{g}-1}$ lie on the distinct II-cycles. □

3. The forcing number for bipartite graphs

In this section, we consider only bipartite graphs with a perfect matching. For convenience, a bipartite graph G has bipartition (W, B), W is the vertex set colored white and B black. Let $\mathcal{M}$ be the set of all perfect matchings of G. A vertex $u \in V(G)$ is a *neighbor* of $v \in V(G)$ if u is adjacent to v. All neighbors of v form its *neighborhood*, denoted by $N(v)$, and define $N[v] = N(v) \cup \{v\}$. More generally for $T \subset V(G)$, the neighborhood of T is defined as $N(T) := (\cup_{v \in T} N(v)) \backslash T$ and $N[T] = N(T) \cup T$.

Define functions α and β on $E(G)$: for any edge $e = wb \in E(G)$ with $w \in W$ and $b \in B$, $\alpha(e) = w$ and $\beta(e) = b$. Given an $M \in \mathcal{M}$, $S \subset M$ and $u \in V(G) \backslash V(S)$, we say S *forces* u if $|N(u) \backslash V(S)| = 1$. In particular, S *W-forces* (resp. *B-forces*) an edge e if $\alpha(e)$ (resp. $\beta(e)$) is forced by S. If there exists a sequence of edges $e_1, e_2, \ldots, e_k$ and a sequence of edge sets $S = S_0, S_1, S_2, \ldots, S_k$ such that $S_i = S_{i-1} \cup \{e_i\}$ and S_{i-1} W-forces (resp. B-forces) e_i $(i = 1, 2, \ldots, k)$, then we say S *W-forces* (resp. *B-forces*) the set S_k.

Lemma 3.1. [16]. S W-forces M if and only if S forces M. □

Let $|M| = n$. Assign an ordering to the edges in M: $e_n > e_{n-1} > \cdots > e_1$. Let $b_n > b_{n-1} > \cdots > b_1$ be the corresponding ordering of the vertices in B, where $b_i = \beta(e_i)$ $(1 \leqslant i \leqslant n)$. For a vertex $b \in B$, b *leads* $N(w)$ if b is the largest vertex among all neighbors of $w \in W$ in the ordering of B; b is called a *leading vertex* if such a $w \in W$ exists, and *trailing vertex* otherwise. Let $S_i = \{e_1, e_2, \ldots, e_i\}$ and $B_i := \beta(S_i) = \{b_1, b_2, \ldots, b_i\}$. Put $\bar{B}_i := \{b_n, \ldots, b_{i+1}\}$.

Lemma 3.2. [16]. If S_i W-forces e_{i+1}, then b_{i+1} leads $N(w_{i+1})$ where $w_{i+1} = \alpha(e_{i+1})$. □

For $T \subset V(G)$, the *excess* of T is defined as $\epsilon(T) = |N(T)| - |T|$. The maximum excess of an ordering $b_n > b_{n-1} > \cdots > b_1$ of B is the maximum value in all $\epsilon(\bar{B}_i)$ $(1 \leqslant i \leqslant n-1)$. The *excess of* b_i is defined to be $\epsilon(b_i) = \epsilon(\bar{B}_{i-1}) - \epsilon(\bar{B}_i)$. We call b_i an *m-excess vertex*, simply *m-ex*, if $\epsilon(b_i) = m$. Riddle gave the following lower bound for $f(G)$:

Lemma 3.3. [16]. $f(G)$ is bounded below by the smallest possible maximum excess for all orderings of B. □

According to the definition of leading vertex and trailing vertex, we have the following lemma:

Lemma 3.4. Let $b_n > b_{n-1} > \cdots > b_i > \cdots > b_1$ be an ordering of B. Then:

(a) the following statements are equivalent:
1. b_i is a leading vertex; 2. $|N(\bar{B}_{i-1})|-|N(\bar{B}_i)| \geqslant 1$; 3. $\epsilon(\bar{B}_{i-1}) \geqslant \epsilon(\bar{B}_i)$;
4. $\epsilon(b_i) \geqslant 0$.

(b) the following statements are equivalent:
1. b_i is a trailing vertex; 2. $|N(\bar{B}_{i-1})| = |N(\bar{B}_i)|$; 3. $\epsilon(\bar{B}_{i-1}) = \epsilon(\bar{B}_i)-1$;
4. $\epsilon(b_i) = -1$.

□

Definition 3.5. An ordering $b_n > b_{n-1} > \cdots > b_1$ of B is *canonical* if its smallest leading vertex is larger than the largest trailing vertex; *non-canonical*, otherwise. □

By lemma 3.4, we have:

Lemma 3.6. The maximum excess of a canonical ordering of B is equal to the number of trailing vertices. □

Lemma 3.7. Let M be a perfect matching of G. If S is a minimum forcing set of M, then there exists a canonical ordering of B such that $\beta(S)$ is the set of trailing vertices and $B\backslash\beta(S)$ is the set of leading vertices.

Proof. Let $|M| = n$ and $S = \{e_1, e_2, \ldots, e_k\} \subseteq M$, the minimum forcing set of M. By Lemma 3.1, we have S W-forces M. Let $S_0 = S$. Then there exists edge $e_{k+j} \in M$ for any $j \in \{1, \ldots, n-k\}$ and edge set $S_j = S_{j-1} \cup \{e_{k+j}\}$ such that S_{j-1} W-forces e_{k+j} $(j = 1, 2, \ldots, n-k)$ and $S_{n-k} = M$. Since $\beta(e_{k+j}) = b_{k+j}$ $(1 \leqslant j \leqslant n-k)$ and lemma 3.2, b_{k+j} $(1 \leqslant j \leqslant n-k)$ is a leading vertex of the ordering of B: $b_n > b_{n-1} > \cdots > b_{k+1} > b_k > \cdots > b_1$. Then $B\backslash\beta(S)$ is the set of leading vertices.

In the following, we want to show that $b_i \in \beta(S)$ $(1 \leqslant i \leqslant k)$ is a trailing vertex. If not, suppose b_i is a leading vertex, then b_i leads a set $N(w_j)$. If $j > k$, then $b_i \geqslant b_j > b_k$ since b_i leads $N(w_j)$ and $w_j b_j = e_j \in M$. Therefore, $i > k$, a contradiction. So $j \leqslant k$. Since $N(w_j)\backslash\{b_j\} \subseteq \beta(S\backslash\{e_j\})$, then $S\backslash\{e_j\}W$-forces e_j and further W-forces M. Hence $|S\backslash\{e_j\}| < |S|$ which contradicts the minimality of $|S|$. Hence $\beta(S)$ is the set of trailing vertices.

Obviously, the ordering $b_n > b_{n-1} > \cdots > b_{k+1} > b_k > \cdots > b_1$ is canonical. □

For any perfect matching M of G, lemma 3.7 implies there exists a canonical ordering of B with $f(G, M)$ trailing vertices.

Theorem 3.8. $f(G)$ is bounded below by the minimum trailing vertex number over all canonical orderings of B. □

Let $b_n > b_{n-1} > \cdots > b_1$ be an ordering of B and $T \subseteq B$, $\bar{T} = B \backslash T$. A vertex $b \in \bar{T}$ is called *forced vertex* of $N[T]$ (also *forced vertex* of T) if $N(b) \subset N[T]$ or $|N(b) \cap (W \backslash N[T])| = 1$. An edge $e \in E(G)$ is *B-forced* by $N[T]$ (also *B-forced* by T) if $\beta(e) \in \bar{T}$, $\alpha(e) \in W \backslash N[T]$ and $N(\beta(e)) \cap (W \backslash N[T]) = \{\alpha(e)\}$. If there exists a sequence of edges $e_1, e_2, \ldots, e_k$ and a sequence of vertex sets $T_0(= T), T_1, T_2, \ldots, T_k$ such that $T_i = T_{i-1} \cup \beta(e_i)$ and T_{i-1} B-forces e_i $(i = 1, 2, \ldots, k)$, then we say the edge set $\{e_1, e_2, \ldots, e_k\}$ is *B-forced* by T. Let S be the maximum B-forced edge set of T and V', the set of all forced vertices of $N[T] \cup \alpha(S) \cup \beta(S)$. Then $N[T] \cup \alpha(S) \cup \beta(S) \cup V'$ is called the *forced domain* of T, denoted by $D(T)$. The forced domain of b_i is defined to be $D(b_i) = D(N[\bar{B}_{i-1}])$. A vertex b_i is called *key vertex* of the ordering if $D(b_{i+1}) \subsetneq D(b_i)$.

Lemma 3.9. Let $b_n > b_{n-1} > \cdots > b_1$ be a canonical ordering of B and $b_{j_1} > b_{j_2} > \cdots > b_{j_l}$, all key vertices of B. Then the maximum excess of the ordering is no less than $\sum_{i=1}^{l} \epsilon(b_{j_i})$.

Proof. Let b_{j_i} be any key vertex of the ordering of B. Then $D(b_{j_i+1}) \subsetneq D(b_{j_i})$. If $\epsilon(b_{j_i}) \leqslant 0$, then $|N(b_{j_i}) \backslash N(\bar{B}_{j_i})| \leqslant 1$. Hence $N[b_{j_i}] \subset D(N[\bar{B}_{j_i}])$ and $D(b_{j_i+1}) = D(N[\bar{B}_{j_i}]) = D(N[\bar{B}_{j_i}] \cup N[b_{j_i}]) = D(N[\bar{B}_{j_i-1}]) = D(b_{j_i})$, a contradiction. So $\epsilon(b_{j_i}) \geqslant 1$.

Since $b_n > b_{n-1} > \cdots > b_1$ is a canonical ordering of B, let b_{k+1} be the smallest leading vertex and then $j_l \geqslant k+1$. By lemma 3.4, $\epsilon(b_i) \geqslant 0$ for $i \geqslant k+1$. Hence $\epsilon(\bar{B}_k)$ is the maximum excess of the ordering and satisfies

$$\epsilon(\bar{B}_k) = \sum_{i>k} \epsilon(b_i) = \sum_{i=1}^{l} \epsilon(b_{j_i}) + \sum_{i>k, i \neq j_1, \ldots, j_l} \epsilon(b_i) \geqslant \sum_{i=1}^{l} \epsilon(b_{j_i}).$$

□

4. The forcing number for toroidal polyhexes

Let $T \subset B$. We say T is *full* in yth layer (or I-column) L if $V(L) \subset T \cup N(T)$ and T *touches* L if $V(L) \cap N[T] \neq \emptyset$ and $V(L) \not\subseteq N[T]$. For any toroidal polyhex $H(p, q, t)$, it has at least three perfect matchings: $M_1 = \{e | e \text{ is vertical in } H(p, q, t)\}$, $M_2 = \{b_{i,j} w_{i,j} | i \in \mathbb{Z}_p, j \in \mathbb{Z}_q\}$, and $M_3 = \{w_{i,j} b_{i+1,j} | i \in \mathbb{Z}_p, j \in \mathbb{Z}_q\}$. Hence $f(H(p, q, t)) \geqslant 1$.

Theorem 4.1. Let $H(p,q,t)$ be a toroidal polyhex. Then $f(H(p,q,t)) \geqslant \min\{p,q\}$.

Proof. If $\min\{p,q\} = 1$, the assertion holds. So, in the following, we suppose $\min\{p,q\} \geqslant 2$.

Let $b_{pq} > b_{pq-1} > \cdots > b_1$ be any ordering of B. Then it suffices to prove there exists $i \in \{1,\ldots,pq\}$ satisfying $\epsilon(\bar{B}_i) \geqslant \min\{p,q\}$ by lemma 3.3. For any ordering of B, let j be the largest one in $\{1,\ldots,pq\}$ such that $\bar{B}_j$ is full in either a yth layer or a I-column. Then we have following cases:

Case 1: If $\bar{B}_j$ is full in a yth layer but not full in any I-column. Hence $\bar{B}_j$ touches every I-column I_k $(0 \leqslant k \leqslant p-1)$. Then $|N(\bar{B}_j \cap V(I_k))| - |\bar{B}_j \cap V(I_k)| \geqslant 1$. Therefore,

$$\epsilon(\bar{B}_j) = |N(\bar{B}_j)| - |\bar{B}_j| = \sum_{k=0}^{p-1}(|N(\bar{B}_j \cap V(I_k))| - |\bar{B}_j \cap V(I_k)|) \geqslant p$$

since $(\bar{B}_j \cap V(I_i)) \cap (\bar{B}_j \cap V(I_r)) = \emptyset$ for $i \neq r$.

Case 2: If $\bar{B}_j$ is full in a I-column but not full in any yth layer. Hence $\bar{B}_j$ touches every layer L_k $(0 \leqslant k \leqslant q-1)$. Then $|N(\bar{B}_j \cap V(L_k))| - |\bar{B}_j \cap V(L_k)| \geqslant 1$. Therefore,

$$\epsilon(\bar{B}_j) = |N(\bar{B}_j)| - |\bar{B}_j| = \sum_{k=0}^{q-1}(|N(\bar{B}_j \cap V(L_k))| - |\bar{B}_j \cap V(L_k)|) \geqslant q$$

since $(\bar{B}_j \cap V(L_i)) \cap (\bar{B}_j \cap V(L_r)) = \emptyset$ for $i \neq r$.

Case 3: If $\bar{B}_j$ is full in a I-column and a yth layer simultaneously. Then $\bar{B}_{j+1}$ touches every I-column and every yth layer. According to cases 1 and 2, $\epsilon(\bar{B}_{j+1}) \geqslant \max\{p,q\}$.

Combining cases 1–3, we have $f(H(p,q,t)) \geqslant \epsilon(\bar{B}_i) \geqslant \min\{p,q\}$ for some $i \in \{1,\ldots,pq\}$ and complete the proof. □

Theorem 4.1 gives a lower bound for the forcing number of toroidal polyhex and it is sharp for infinitely many toroidal polyhexes, implied by the following theorem.

Theorem 4.2.

$$f(H(p,q,t)) = \begin{cases} p, & \text{if } p \leqslant q, \\ q, & \text{if } p > q \text{ and } t = 0, p-1, \ldots, p-q. \end{cases}$$

Proof. If $p \leqslant q$. Since $H(1,1,0)$ has only two vertices and three edges, then $f(H(1,1,0)) = 1$. So suppose $2 \leqslant q$ and let $S = \{b_{j,0}w_{j-1,1} | 0 \leqslant j \leqslant p-1\}$. Then S forces a perfect matching M_1. Hence $f(H(p,q,t), M_1) \leqslant |S| = q$ for $1 \leqslant p \leqslant q$. By theorem 4.1, we have $f(H(p,q,t)) = p$.

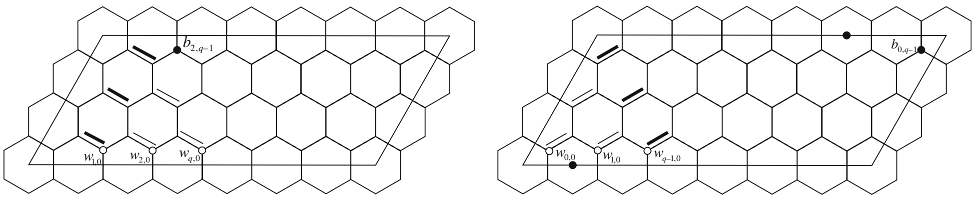

Figure 2. Toroidal polyhex $H(7, 3, t)$ and illustration for proof of theorem 4.2.

If $p > q$. Let $S = \{b_{1,j}w_{1,j} | 0 \leqslant j \leqslant q-1\}$. Clearly, S forces $E = \{b_{i,j}w_{i,j} | 1 \leqslant i \leqslant q, j = q-i\}$. So S forces M_2 if and only if $S \cup E$ forces edge $b_{2,q-1}w_{2,q-1}$, equivalently $w_{1-t,0} \in N(b_{2,q-1}) \cap \{w_{1,0}, \ldots, w_{q,0}\}$ (see figure 2 (left)), just $1 \leqslant p+1-t \leqslant q$ and further $p-q+1 \leqslant t \leqslant p$. Therefore, $f(H(p,q,t)) \leqslant |S| = q$ for $t = 0, p-1, \ldots, p-q+1$. For $t = p-q$, let $S = \{w_{i,j}b_{i+1,j} | 0 \leqslant i \leqslant q-1, j = q-1-i\}$. Then S forces M_3 (see figure 2 (right)). Hence $f(H(p,q,t)) \leqslant |S| = q$ for $t = p-q$. By theorem 4.1, we have $f(H(p,q,t)) = q$ for $p > q$ and $t = 0, p-1, \ldots, p-q$. □

Theorem 4.2 gives the forcing numbers of partial toroidal polyhexes. For the toroidal polyhex $H(p,q,t)$ with $p > q \geqslant 1$ and $1 \leqslant t \leqslant p-q-1$, it becomes a little complicated to give its forcing number. Let H denote a toroidal polyhex $H(p,q,t)$ for convenience. For a vertex set $S \subset V(H)$, $H[S]$ is the subgraph induced by S in H.

A *triangle* T on H is defined to be an equilateral triangle whose corners lie on the centers of three hexagons $h_{x_1,y}$, $h_{x_2,y}$, and $h_{x_3,y'}$ such that $w_{x_1,y}$ and $w_{x_3,y'}$ are on the same I-cycle and $w_{x_2-1,y}$ and $w_{x_3-1,y'}$ are on the same II-cycle, the side length of T is $|x_2 - x_1|$, denoted by $\delta(T)$. For convenience, we use a hexagon notation to denote its center, then $T = h_{x_1,y}h_{x_2,y}h_{x_3,y'}$. A triangle T is *maximum* on H if $\delta(T)$ is largest among all triangles. The triangle $h_{0,0}h_{k,0}h_{i,j}$ ($i = p - st$ and $j = k - sq$ with $s \geqslant 0, i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_q$) is also called *normal* triangle, denoted by $\triangle_k$. By the automorphism ϕ_{tb} and ϕ_{rl}, every triangle is isomorphic to a normal triangle. According to the representation of H in the plane, $\triangle_k$ consists of s trapeziums $P_{i+1} = h_{p-it,0}h_{k-i(q+t),0}h_{k-(i+1)(q+t),0}h_{p-(i+1)t,0}$ with $0 \leqslant i \leqslant s-1$ and a small triangle $P_{s+1} = h_{p-st,0}h_{k-s(q+t),0}h_{p-st,k-sq}$. For example, the $\triangle_5$ in $H(11,3,3)$ consists of a trapezium $P_1 = h_{0,0}h_{5,0}h_{10,0}h_{8,0}$ and a triangle $P_2 = h_{8,0}h_{10,0}h_{8,2}$ (see figure 3). Simply, we also use a triangle to denote the vertex set consisting of all vertices lying in it, for example, $\triangle_2 = N[b_{1,0}]$. For a normal triangle $\triangle_i$, define $\bar{\triangle}_i = V(H) - \triangle_i$.

Let T be a triangle, II_x is *adjacent* to T if $V(II_x) \cap T = \emptyset$ and $V(II_x) \cap N(T) \neq \emptyset$. A vertex $b \in V(II_x)$ is *II-adjacent* to T if $b \in N(T)$ and $V(II_x)$ is adjacent to T (II_5 and II_{10} are adjacent to $\triangle_5$ and all black vertices on II_5 are II-adjacent to $\triangle_k$ in figure 3), let $N_{II}(T)$ be the vertex set consisting all II-adjacent vertices together with their neighbors in all II-columns adjacent to T. If a normal triangle $\triangle_i$ satisfies $|\triangle_i \cap N(b)| \leqslant 1$ for any $b \in \bar{\triangle}_i$, then

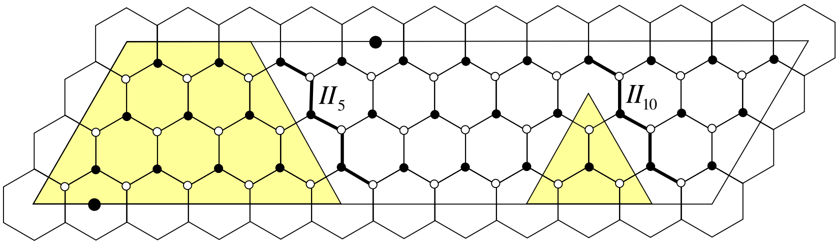

Figure 3. Toroidal polyhex $H(11, 3, 3)$ and the normal triangle $\triangle_5$.

$\triangle_{i+1} = \triangle_i \cup N_{II}(\triangle_i)$. The process from $\triangle_i$ to $\triangle_{i+1}$ is called *triangle extension*. We continue the triangle extension and stop at $\triangle_k$ which satisfies there exists a vertex $b \in \bar{\triangle}_k$ such that $|N(b) \cap \triangle_k| = 2$, the $\triangle_k$ is called *characteristic triangle* of H.

Lemma 4.3. Let $\triangle_k$ be the characteristic triangle of H. Then for any $i < k$, the normal triangle $\triangle_i$ satisfies $D(\triangle_i) = \triangle_i$.

Proof. For any $\triangle_i$ $(i < k)$, $|N(v) \cap \triangle_i| \leqslant 1$ for any vertex $b \in \bar{\triangle}_i$ since $\triangle_k$ is a characteristic triangle. Hence $\triangle_i$ does not B-force v. Immediately, we have $D(\triangle_i) = \triangle_i$. □

Theorem 4.4. Let $\triangle_k$ consist of s trapezia P_{i+1} $(0 \leqslant i \leqslant s-1)$ and one triangle P_{s+1}. Then $\triangle_k$ is the characteristic triangle if and only if one of following cases appears:

(1) there exists l $(0 \leqslant l \leqslant s)$ such that P_1 and P_{l+1} have the same corner $h_{k,0}$;

(2) there exists l $(0 \leqslant l \leqslant s)$ such that P_1 and P_{l+1} have the same corner $h_{0,0}$;

(3) the corner $h_{p-st,k-sq}$ of P_{s+1} coincides with $h_{x,0}$ where $0 \leqslant x \leqslant k$.

Proof. *Sufficiency:* It suffices to prove there exists $b \in \bar{\triangle}_k$ such that $|N(b) \cap \triangle_k| = 2$. If (1) holds, let $b = b_{k,0} \in \bar{\triangle}_k$. Since $w_{k-1,0} \in P_1$ and $w_{k,0} \in P_{l+1}$, $|N(b_{k,0}) \cap \triangle_k| = 2$. If (2) holds, let $b = b_{0,0} \in \bar{\triangle}_k$. Hence $|N(b_{0,0}) \cap \triangle_k| = 2$ since $w_{p-1,0} \in P_{l+1}$, $w_{0,0} \in P_1$. If (3) holds, then let $b = b_{x+t+1,q-1} \in \bar{\triangle}_k$ if $x < k$ and $b = b_{x+t,q-1} \in \bar{\triangle}_k$ if $x = k$. Then the assertion holds since $w_{x,0} \in P_1$, $w_{x+t,q-1} \in P_{s+1}$ for $x < k$ and $w_{x-1,0} \in P_1$, $w_{x+t,q-1} \in P_{s+1}$ for $x = k$.

Necessary: Since $\triangle_k$ is the characteristic triangle, there exists $b \in \bar{\triangle}_k$ such that $|N(b) \cap \triangle_k| = 2$. Hence $b \in N(\triangle_k)$. For a vertex $b \in \triangle_k \cap B$, we have

$N(b) \subset \triangle_k$. So $b \in B \cap \bar{\triangle}_k$, say $b = b_{i,j}$. Then $N(b_{i,j}) = \{w_{i-1,j}, w_{i,j}, w_{x,y}\}$ where $x = i-1, y = j+1$ if $j \neq q-1$ and $x = i-t-1, y = 0$ if $j = q-1$.

Case 1: If $w_{i-1,j} \in P_{l_1}$ and $w_{i,j} \in P_{l_2}$. Then the center $h_{i,j}$ is a point in the intersection of P_{l_1} and P_{l_2}. If $j \neq 0$, then the vertex $b_{i,j-1} \notin \triangle_{k-1}$ satisfies $|N(b_{i,j-1}) \cap \triangle_{k-1}| = 2$ since $w_{i-1,j-1}, w_{i,j-1} \in \triangle_{k-1}$, then $b_{i,j-1} \in D(\triangle_{k-1})$ which contradicts $D(\triangle_{k-1}) = \triangle_{k-1}$ by lemma 4.3, so $j = 0$. If $\min\{l_1, l_2\} \neq 1$, then the vertex $b_{i+t,q-1} \notin \triangle_{k-1}$ satisfies $|N(b_{i+t,q-1}) \cap \triangle_{k-1}| = 2$ since $w_{i+t,q-1}, w_{i+t-1,q-1} \in \triangle_{k-1}$, then $b_{i+t,q} \in D(\triangle_{k-1})$, a contradiction. Therefore, we have $\min\{l_1, l_2\} = 1$ and $j = 0$. If $l_1 = 1$, then (1) appears. If $l_2 = 1$, then (2) appears.

Case 2: If $w_{i-1,j} \in P_{l_1}$ and $w_{x,y} \in P_{l_2}$ or $w_{i,j} \in P_{l_1}$ and $w_{x,y} \in P_{l_2}$. If $l_2 \neq 1$ or $y \neq 0$, we have $w_{x,y}b_{i,j} \in E(H[\triangle_k])$ which contradicts $b_{i,j} \in \bar{\triangle}_k$, so $l_2 = 1$ and $y \neq 0$, just $j = q-1$. If P_{l_1} is a trapezium, then $b_{i-1,q-1} \in P_{l_1}$. Since $w_{x,0} \in P_1$, then $0 \leqslant x \leqslant k$. If $x > 0$, then $w_{x-1,0}, w_{i-2,q-1} \in \triangle_k$ and further $|N(b_{i-1,q-1}) \cap \triangle_{k-1}| = 2$, then $b_{i-1,q-1} \in D(\triangle_{k-1})$ which contradicts lemma 4.3. So $x = 0$, then P_{l_1+1} and P_1 have the same corner $h_{0,0}$, (2) appears. If P_{l_1} is a triangle, then $l_1 = s+1$, further the corner $h_{p-st,k-sq} = h_{x+t+1,q-1}$ of P_{s+1} coincides with $h_{x,0}$ $(0 \leqslant x \leqslant k)$, (3) appears. □

According to the isomorphism ϕ_{tb} and ϕ_{rl} of H, theorem 4.4 and its proof imply the characteristic triangle is, in fact, a maximum triangle on the toroidal polyhex and every maximum triangle is also isomorphic to the characteristic triangle.

Lemma 4.5. Let $\triangle_k$ be the characteristic triangle of H and every II-cycle (resp. I-cycle) of H has g (resp. g') II-columns (resp. I-columns). Then $k \geqslant \frac{p}{g}$ if one of cases (1)–(3) with $x \neq k$ in theorem 4.4 appears and $k \geqslant \frac{p}{g'}$ if case (3) with $x = k$ in theorem 4.4 appears.

Proof. *Case 1:* If P_1 and P_{l+1} $(1 \leqslant l \leqslant s)$ have the same corner $h_{k,0}$. Then $k \equiv p - lt \pmod p$. Hence there exists $\lambda \in \mathbb{Z}^+$ such that $k = \lambda p - lt$. Since a II-cycle contains g II-columns, by lemma 2.1 we have $g(q+t) \equiv 0 \pmod p$, further $g[p-(q+t)] \equiv 0 \pmod p$. So there exists $\mu \in \mathbb{Z}$ such that $g[p-(q+t)] = \mu p$, further $(g-\mu)p = g(q+t)$.

Since $k - sq \geqslant 1$, $k \geqslant sq + 1 > lq$. Hence

$$\begin{aligned} g(k-lq) &= g[(\lambda p - lt) - lq] = g[\lambda p - l(q+t)] = g\lambda p - g(q+t)l \\ &= [g(\lambda - l) + \mu l]p. \end{aligned}$$

Then $g(k-lq) \geqslant p$ since $g(k-lq) > 0$. Therefore, $k > \frac{p}{g}$.

Case 2: If P_1 and P_{l+1} $(1 \leqslant l \leqslant s)$ have the same corner $h_{0,0}$. Then $k - l(q+t) \equiv 0 \pmod p$, further $l(p-q-t)+k \equiv 0 \pmod p$. Let $\gamma \in \mathbb{Z}$ satisfy $l(p-q-t)+k = \gamma p$. Then $(l-\gamma)p + k = l(q+t)$, further $g(l-\gamma)p + gk = gl(q+t)$.

Hence $gk = (l-\gamma)gp + lg(q+t)$. By lemma 2.1, $g(q+t) \equiv 0 \pmod{p}$. Therefore, $gk \equiv 0 \pmod{p}$. Clearly, $gk > 0$. Hence $gk > p$, just $k > \frac{p}{g}$.

Case 3: If $h_{p-st,k+1-sq} = h_{x,0}$ for $0 \leqslant x \leqslant k$.

Subcase 3.1: If $0 \leqslant x \leqslant k-1$. According to $h_{p-st,k-sq} = h_{x,0}$, we have $k - sq - q = 0$ and $p - st - t = x$. Further, $k = (1+s)q$ and $0 \leqslant p - (1+s)t < k \pmod{p}$. Then there exists $\eta \in \mathbb{Z}$ such that $0 \leqslant \eta p - (1+s)t < k$.

By lemma 2.1, $g(q+t) \equiv 0 \pmod{p}$. So there exists $\theta \in \mathbb{Z}$ such that $\theta p = g(q+t)$, then $gq = \theta p - gt$. Hence

$$gk = g(1+s)q = (1+s)(\theta p - gt) = (1+s)\theta p - (1+s)gt$$

and

$$gk > g[\eta p - (1+s)t] = g\eta p - (1+s)gt \geqslant 0.$$

Therefore $(1+s)\theta > g\eta$, further $(1+s)\theta \geqslant 1 + g\eta$. Then $gk = (1+s)\theta p - (1+s)gt \geqslant (1+g\eta)p - (1+s)gt = p + [g\eta p - (1+s)gt] \geqslant p$. So $k \geqslant \frac{p}{g}$.

Subcase 3.2: If $x = k$, then $k \equiv p - (s+1)t \pmod{p}$. Hence there exists $\eta \in \mathbb{Z}^+$ such that $k = \eta p - (s+1)t$. Then $g'k = \eta g' p - (s+1)g't$. By lemma 2.1, $g't \equiv 0 \pmod{p}$. There exists $\theta \in \mathbb{Z}$ such that $g'k = [\eta g' - (s+1)\theta]p$. Since $g'k > 0$, hence $g'k \geqslant p$. So $k \geqslant \frac{p}{g'}$. □

Let $G \subset H$, an edge $e \in E(G)$ is called a *pendant edge* if $\beta(e)$ is a 1-degree vertex. Clearly, a pendant edge e is B-forced by $V(H - G)$.

Lemma 4.6. Let $\triangle_k$ be the characteristic triangle of toroidal polyhex H. Then $D(\triangle_k) = V(H)$.

Proof. Let $\triangle_k$ consist of s trapeziums P_{l+1} $(0 \leqslant l \leqslant s-1)$ and a triangle P_{s+1}, $H_0 := H[\bar{\triangle}_k]$ is the subgraph of H induced by $\bar{\triangle}_k$ (see Figure 4).

Case 1: There exists $0 \leqslant l \leqslant s$ such that P_1 and P_{l+1} have the same corner $h_{k,0}$. Let $S^1 = E(H_0) \cap M_1$. Then we have the following claim:

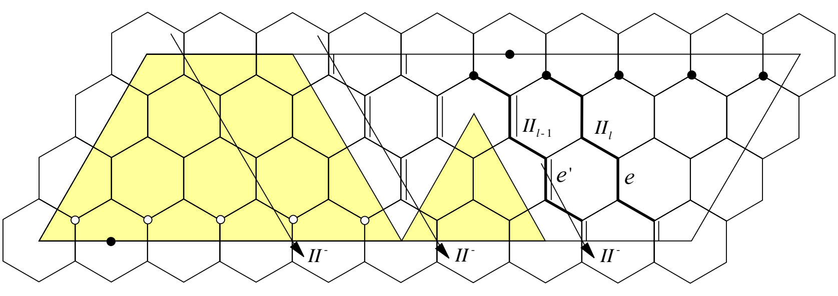

Figure 4. Illustration for case 1 in proof of lemma 4.6.

Claim 1: S^1 is B-forced by $\triangle_k$.

Proof. Clearly, $b_{k,0}w_{k-1,1}$ is a pendant edge of H_0 and is forced by $\triangle_k$. Let $e_1 = b_{k,0}w_{k-1,1}$ and $H_1 = H_0 - \{b_{k,0}, w_{k-1,1}\}$. Define S_i as the vertical pendant edge set of H_i and $H_{i+1} := H_i - V(S_i)$, $i = 0, 1, 2, \ldots$

Suppose to the contrary that there exist edges in S^1 not B-forced by $\triangle_k$, equivalently, there exists $H_m \subset H_0$ such that $E(H_m)$ contains no pendant vertical edge and $E(H_m) \cap S^1 \neq \emptyset$. Choose one edge $e \in E(H_m) \cap S^1$ such that $e \in II_l$ and l is minimal. By the minimality of l, for any $e' \in E(II_{l-1})$, either $e' \in E(H[\triangle_k])$ or e' is a pendant edge of some H_i with $i < m$. Let R_{l-1} and R_l be the II-cycle containing II_{l-1} and II_l, respectively. By lemma 4.5, every II-cycle contains at least one edge in $E' = \{w_{j,0}b_{j+t+1,q-1} | 0 \leqslant j \leqslant k-1\}$. Hence, all vertical edges in $E(R_{l-1})$ starting from e' along II^--direction and stoping at some edge in $E' \cap E(R_{l-1})$ are not in $E(H_m)$. Since e is not a pendent edge, therefore all vertical edge in $E(R_l)$ starting from e along II^--direction and stoping at some edge in $E' \cap E(R_l)$ belong to $E(H_m)$; If not, $E(H_m)$ contains vertical pendant edge which contradicts the supposition. But $E' \cap E(H_0) = \emptyset$, which contradicts $H_m \subset H_0$ and $E' \cap E(R_l) \cap E(H_m) \neq \emptyset$. The contradiction implies claim 1.

Since $S^1 = E(H_0) \cap M_1$, then $H - (V(S^1) \cup \triangle_k)$ consists of k isolated vertices: $b_{t+i,q-1}$ $(1 \leqslant i \leqslant k)$. Then $N(b_{t+i,q-1}) \subset \triangle_k \cup V(S^1)$, so $b_{t+i,q-1} \in D(\triangle_k)$ since S^1 is B-forced by $\triangle_k$. Further, $D(\triangle_k) = V(H)$.

Case 2: There exists $0 \leqslant l \leqslant s$ such that P_1 and P_{l+1} have the same corner $h_{0,0}$. Let $S^2 = E(H_0) \cap M_2$. Then we have following claim (see Figure 5):

Claim 2: S^2 is B-forced by $\triangle_k$.

Proof. Since $P_l = h_{p-(l-1)t,0}h_{k-(l-1)(q+t),0}h_{k-l(q+t),0}h_{p-lt,0}$, hence $k - l(q+t) \equiv 0 \pmod p$ and further $k - (l-1)(q+t) - q \equiv t \pmod p$, which implies the black vertex $b_{t+i,q-1}$ $(1 \leqslant i \leqslant k)$ is adjacent to $w_{i-1,0}$ belonging to P_1. Hence, $b_{t+1,q-1}w_{t+1,q-1}$ is a pendant edge of H_0 since $w_{t,q-1} \in P_l \cap N(b_{t+1,q-1})$ and $w_{0,0} \in P_1 \cap N(b_{t+1,q-1})$. Let $H_1 = H_0 - \{b_{t+1,q-1}, w_{t+1,q-1}\}$. Define $S_i \subset S^2$ is the pendant edge set of H_i and $H_{i+1} := H_i - V(S_i)$, $i = 0, 1, \ldots$

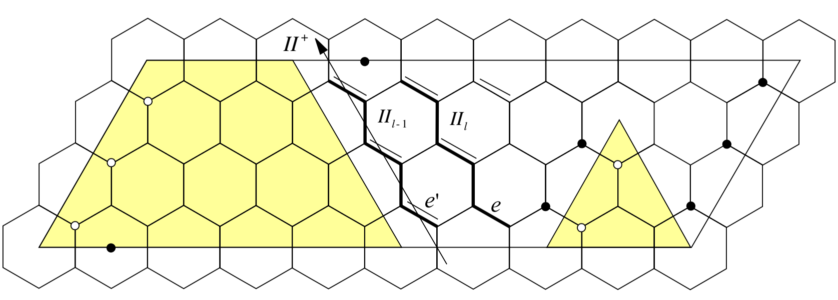

Figure 5. Illustration for case 2 in proof of lemma 4.6.

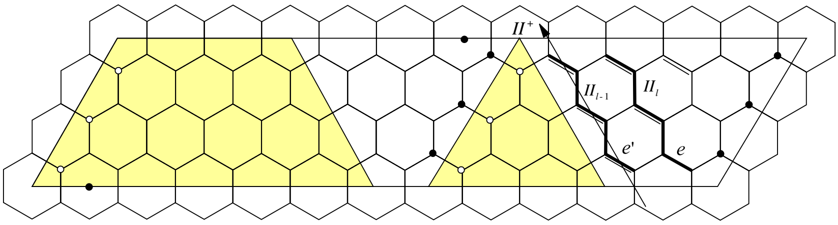

Figure 6. Illustration for subcase 3.1 in proof of lemma 4.6.

Suppose to the contrary that there exist edges in S^2 not B-forced by $\triangle_k$, equivalently, there exists $H_m \subset H_0$ such that every edge in $E(H_m) \cap S^2 \neq \emptyset$ is not a pendant edge of H_m. Choose one edge $e \in E(H_m) \cap S^2$ such that $e \in II_l$ and l is minimal. By the minimality of l, for any $e' \in E(II_{l-1}) \cap M_2$, either $e' \in E(H[\triangle_k])$ or e' is a pendant edge of some H_i with $i < m$. By lemma 4.5, every II-cycle contains at least one edge in $E' = \{w_{j,0}b_{j,0} | 0 \leqslant j \leqslant k-1\}$. Hence, all edges in $E(R_{l-1}) \cap M_2$ starting from e' along II^+-direction and stoping at some edge in $E' \cap E(R_{l-1})$ are not in $E(H_m)$. Since e is not a pendent edge, all vertical edge in $E(R_l)$ starting from e along II^+-direction and stoping at some edge in $E' \cap E(R_l)$ belong to $E(H_m)$; If not, $E(H_m)$ contains pendant edge in S^2 which contradicts the supposition. But $E' \cap E(H_0) = \emptyset$, which contradicts $H_m \subset H_0$ and $E' \cap E(R_l) \cap E(H_m) \neq \emptyset$. The contradiction implies claim 2.

Since S^2 is B-forced by $\triangle_k$, $H_0 - V(S^2)$ has no edges, hence for any vertex in $H_0 - V(S^2)$, its neighbors belongs to $\triangle_k \cup V(S^2)$. Hence $D(\triangle_k) = V(H)$.

Case 3: The corner $h_{p-st,k-sq}$ of P_{s+1} coincides with $h_{x,0}$ where $0 \leqslant x \leqslant k$.

Subcase 3.1: If $x < k$. Then $b_{x+1,q-1}w_{x+1,q-1}$ is a pendant edge of H_0. Let $H_1 = H_0 - \{b_{x+1,q-1}, w_{x+1,q-1}\}$ (see Figure 6).

Further, by the same discussion as that of case 2, we have $E(H_0) \cap M_2$ is B-forced by $\triangle_k$. Since $H_0 - V(E(H_0) \cap M_2)$ has only k isolated vertices, $D(\triangle_k) = V(H)$.

Subcase 3.2: If $x = k$. Then $b_{x,q-1}w_{x-1,q-1}$ is a pendant edge of H_0. Let $H_1 = H_0 - \{b_{x,q-1}, w_{x-1,q-1}\}$.

By the same discussion of subcase 3.1 but changing II-cycle to I-cycle, we have $D(\triangle_k) = V(H)$. □

Lemma 4.7. Let $\triangle_k$ be the characteristic triangle of H. Then $f(H) \leqslant k$.

Proof. It suffices to find a perfect matching M of H such that $f(H, M) \leqslant k$. Let $\triangle_k$ consist of s trapeziums P_{l+1} $(0 \leqslant l \leqslant s-1)$ and a triangle P_{s+1}.

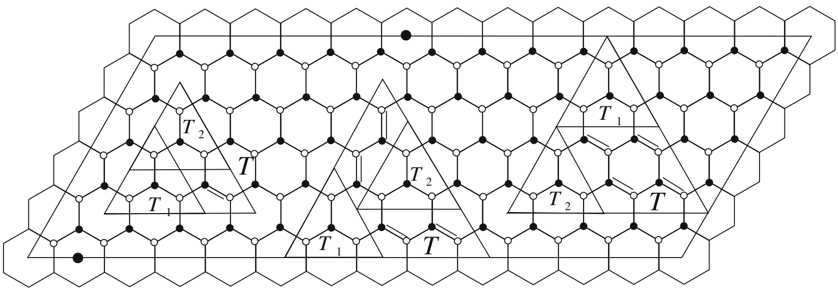

Figure 7. T_1 and T_2, the double edges are B-forced by $T_1 \cup T_2$.

Case 1: If there exists $0 \leqslant l \leqslant s$ such that P_1 and P_{l+1} have the same corner $h_{k,0}$. Let $S = \{w_{i,0}b_{i+t+1,q-1} | 0 \leqslant i \leqslant k-1\}$. Then S forces $E(H[\triangle_k]) \cap M_1$. By lemma 4.6, in this case, $\triangle_k$ forces $E(H[\bar{\triangle}_k]) \cap M_1$. Since $M_1 = S \cup (E(H[\triangle_k]) \cap M_1) \cup (E(H[\bar{\triangle}_k]) \cap M_1)$, we have S forces M_1. Further, $f(H(p,q,t), M_1) \leqslant |S| = k$.

Case 2: If there exists $0 \leqslant l \leqslant s$ such that P_1 and P_{l+1} have the same corner $h_{0,0}$. Let $S = \{b_{p-rt,i}w_{p-rt,i} |\ 0 \leqslant r \leqslant \lfloor \frac{k}{q} \rfloor, 0 \leqslant i \leqslant q-1$ and $r = \lceil \frac{k}{q} \rceil, 0 \leqslant i \leqslant k-(r-1)q-1\}$. Then S forces $E(H[\triangle_k]) \cap M_2$. Since $\triangle_k$ forces $E(H[\bar{\triangle}_k]) \cap M_2$ and $M_2 = S \cup (E(H[\triangle_k]) \cap M_2) \cup (E(H[\bar{\triangle}_k]) \cap M_2)$, we have S forces M_2 and then $f(H(p,q,t), M_2) \leqslant |S| = k$.

Case 3: The corner $h_{p-st,k-sq}$ of P_{s+1} coincides with $h_{x,0}$ where $0 \leqslant x \leqslant k$.

Subcase 3.1: For $0 \leqslant x < k$. Let $S = \{b_{p-rt,i}w_{p-rt,i} |\ 0 \leqslant r \leqslant \lfloor \frac{k}{q} \rfloor, 0 \leqslant i \leqslant q-1$ and $r = \lceil \frac{k}{q} \rceil, 0 \leqslant i \leqslant k-(r-1)q-1\}$. As discussion in case 2, we have S forces M_2 and then $f(H, M_2) \leqslant |S| = k$.

Subcase 3.2: For $x = k$. Let $S = \{b_{k-r(p+t),i}w_{k-r(p+t)-1,i} |\ 0 \leqslant r \leqslant \lfloor \frac{k}{q} \rfloor, 0 \leqslant i \leqslant q-1$ and $r = \lceil \frac{k}{q} \rceil, 0 \leqslant i \leqslant k-(r-1)q-1\}$. Then S forces $E(H[\triangle_k]) \cap M_3$. Since $\triangle_k$ forces $E(H[\bar{\triangle}_k]) \cap M_3$ and $M_3 = S \cup (E(H[\triangle_k]) \cap M_3) \cup (E(H[\bar{\triangle}_k]) \cap M_3)$, we have S forces M_3 and further $f(H, M_3) \leqslant |S| = k$. □

Let triangles T_1 and T_2 satisfy $T_1 = N[b_{x_1,y_1}]$ and $T_2 = N[b_{x_2,y_2}]$. If T_1 and T_2 have a common point, then $D(T_1 \cup T_2)$ is the minimal triangle T such that $T_1 \cup T_2 \subset T$ if $\triangle_k$ satisfies $k > \delta(T)$ (see figure 7). For generality, let T_1 and T_2 be two triangles with $\delta(T_i) < k (i = 1, 2)$. We say T_1 and T_2 are *disjoint* if they have no common point. If T_1 and T_2 have a common point, let T_* be the region of intersection of T_1 and T_2, then $D(T_1 \cup T_2)$ is the minimal triangle containing $T_1 \cup T_2$ if $\delta(T_1) + \delta(T_2) - \delta(T_*) < k$ and $D(T_1 \cup T_2) = V(H)$ if $\delta(T_1) + \delta(T_2) - \delta(T_*) \geqslant k$, where k is the side length of the characteristic triangle of H. We omit the proof here.

Lemma 4.8. Let $\triangle_k$ be the characteristic triangle of H and $b_{pq} > b_{pq-1} > \cdots > b_1$ be any canonical ordering of B whose key vertices are $b_{j_1} > \cdots > b_{j_l}$. Then $\sum_{i=1}^{l} \epsilon(b_{j_i}) \geqslant k$.

Proof. Since H is a 3-regular graph and b_{j_i} $(1 \leqslant i \leqslant l)$ is key vertex, we have $1 \leqslant \epsilon(b_{j_i}) \leqslant 2$. Clearly we have $b_{pq} = b_{j_1}$, $\epsilon(b_{j_1}) = 2$ and $D(b_{j_l}) = V(H)$. If $l = 1$, then $D(N[b_{j_1}]) = V(H)$. According to the isomorphism ϕ_{tb} and ϕ_{rl}, let $b_{j_1} = b_{1,0}$. Then $\triangle_2 = N[b_{j_1}]$, so $D(\triangle_2) = V(H)$, hence $\triangle_2$ is a characteristic triangle. Therefore, $k = 2 \leqslant \epsilon(b_{j_1})$ and the assertion holds.

So, in the following, we suppose $l > 1$. Then $D(b_{j_{l-1}}) \subsetneq V(H)$.

Claim: $D(b_{j_i})$ $(1 \leqslant i \leqslant l-1)$ consists of some disjoint triangles T such that $\delta(T) < k$ and $\sum_{b_{j_t \in T}} \epsilon(b_{j_t}) \geqslant \delta(T)$ for $1 \leqslant t \leqslant i$.

Proof. We prove it by induction on i. If $i = 1$, let $b_{j_1} = b_{x,y}$. Then $D(b_{j_1}) = N[b_{x,y}]$. So $D(b_{j_1})$ consists only of one triangle $T = N[b_{x,y}]$ with side length 2. On the other hand, b_{j_1} is the maximum key vertex of the ordering B, so $\epsilon(b_{j_1}) = 2 \geqslant \delta(T)$. Hence the claim holds for $i = 1$.

In the following, we assume claim is true for $i-1$, then $D(b_{j_{i-1}})$ consists of some disjoint triangles $T_1, \ldots, T_r$ and $\sum_{b_{j_t} \in T_m} \epsilon(b_{j_t}) \geqslant \delta(T_m)$ $(1 \leqslant t \leqslant i-1$, $1 \leqslant m \leqslant r)$. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_r\}$. For the key vertex b_{j_i}, let T^0 be the triangle such that $T^0 = N[b_{j_i}]$. If T^0 has no common points with T_m $(1 \leqslant m \leqslant r)$, then $\epsilon(b_{j_i}) = 2$ and claim is true since $\delta(T) = 2$. Without loss of generality, suppose there exists a sequence of triangles $T_{m_1}, \ldots, T_{m_{r_1}} \in \mathcal{T}$ such that $T_{m_{j+1}}$ has a common point with T^j, where T^j is the minimal triangle satisfying $T^{j-1} \cup T_{m_j} \subseteq T^j$, and for every $T' \in \mathcal{T}, T'$ has a common point with T^{r_1} if and only if $T' \subseteq T^{r_1}$. Let $T_*^j = T^{j-1} \cap T_{m_j} (1 \leqslant j \leqslant r_1)$. Then $\delta(T^j) = \delta(T_{m_j}) + \delta(T^{j-1}) - \delta(T_*^j)$ and $\delta(T^{r_1}) < k$, otherwise contradict with $i \leqslant l-1$. Let $\mathcal{T}_m = \{T_{m_1}, T_{m_2}, \ldots, T_{m_{r_1}}\}$. We have

$$\delta(T^{r_1}) = \delta(T^{r_1-1}) + \delta(T_{m_{r_1}}) - \delta(T_*^{r_1}) \leqslant \delta(T^{r_1-1}) + \delta(T_{m_{r_1}}) \leqslant \sum_{j=1}^{r_1} \delta(T_{m_j}) + \delta(T^0) - \delta(T_*^1)$$

$$\leqslant \sum_{j=1}^{r_1} \sum_{b_{j_t} \in T_{m_j}} \epsilon(b_{j_t}) + \epsilon(b_{j_i}) + \sum_{T' \in \mathcal{T} \setminus \mathcal{T}_m \text{ and } T' \subset T^{r_1}} \delta(T') \leqslant \sum_{b_{j_t} \in T^{r_1}} \epsilon(b_{j_t}).$$

Therefore, the claim holds.

In the following, we will prove $\sum_{i=1}^{l} \epsilon(b_{j_i}) \geqslant k$. Suppose that $D(b_{j_{l-1}})$ consists of r disjoint triangles $T_1, \ldots, T_r$. Let $T^0 = N[b_{j_l}]$. Since $1 \leqslant \epsilon(b_{j_l}) \leqslant 2$ and $D(b_{j_l}) = V(H)$, there exists $T_{m_1} (1 \leqslant m_1 \leqslant r)$ such that T_{m_1} has a common point with T^0. Then either $\delta(T_{m_1}) + \delta(T^0) - \delta(T_*^1) \geqslant k$ where $T_*^1 = T^0 \cap T_{m_1}$ or there is a minimal triangle T^1 such that $T^0 \cup T_{m_1} \subset T^1$ and $\delta(T^1) = \delta(T_{m_1}) + \delta(T^0) - \delta(T_*^1) < k$.

If the former holds, we have

$$k \leqslant \delta(T_{m_1}) + \delta(T^0) - \delta(T_*^1) \leqslant \sum_{b_{j_i} \in T_{m_1}} \epsilon(b_{j_i}) + \epsilon(b_{j_l}) \leqslant \sum_{i=1}^{l} \epsilon(b_{j_i}),$$

the assertion holds. If the latter holds, without loss of generality, suppose there exists a sequence of triangles $T_{m_1}, T_{m_2}, \ldots, T_{m_{r_1}}$ $(1 \leqslant m_i \leqslant r$ for $i = 1, 2, \ldots, r_1)$ and triangles $T^0, T^1, T^2, \ldots, T^{r_1}$, such that T^j has a common point with $T_{m_{j+1}}$ and is minimal subject to $T^{j-1} \cup T_{m_j} \subseteq T^j$, and every $T_i (1 \leqslant i \leqslant r)$ has a common point with T^{r_1} if and only if $T_i \subseteq T^{r_1}$. Let $T_*^j = T^{j-1} \cap T_{m_j}$. Then $\delta(T^{r_1}) = \delta(T^{r_1 - 1}) + \delta(T_{m_{r_1}}) - \delta(T_*^{r_1}) \geqslant k$ by $D(b_{j_l}) = V(H)$. According to the claim, we have

$$\begin{aligned} k &\leqslant \delta(T^{r_1-1}) + \delta(T_{m_{r_1}}) - \delta(T_*^{r_1}) \leqslant \delta(T^{r_1-1}) + \delta(T_{m_{r_1}}) \\ &\leqslant \sum_{j=1}^{r_1} \delta(T_{m_j}) + \delta(T^0) - \delta(T_*^1) \leqslant \sum_{j=1}^{r_1} \sum_{b_{j_i} \in T_{m_j}} \epsilon(b_{j_i}) + \epsilon(b_{j_l}) \leqslant \sum_{i=1}^{l} \epsilon(b_{j_i}). \end{aligned}$$

The assertion holds. □

Theorem 4.9. Let $\triangle_k$ be the characteristic triangle of $H(p, q, t)$. Then $f(H(p, q, t)) = k$.

Proof. By lemmas 3.9 and 4.8, we know the smallest possible maximum excess over all canonical orderings of B is no less than k. Hence $f(H(p, q, t)) \geqslant k$ by lemma 3.6 and theorem 3.8. By lemma 4.7, $f(H(p, q, t)) \leqslant k$. So $f(H(p, q, t)) = k$. □

5. An algorithm

We conclude this paper with a fast algorithm to compute $f(H(p, q, t))$ with $p > q \geqslant 1$ and $1 \leqslant t \leqslant p - q - 1$, based on theorem 4.2, which gives the forcing number of a toroidal polyhex $H(p, q, t)$ with $1 \leqslant p \leqslant q$ or $p > q \geqslant 1$ and $t \in \{p - q, p - q + 1, \ldots, p - 1, 0\}$.

According to the triangle extension introduced in section 4 and theorems 4.4 and 4.9, we have the following algorithm of complexity $O(n)$, where n is the number of vertices of $H(p, q, t)$.

Algorithm 5.1. Input: A toroidal polyhex $H(p, q, t)$ with $p > q \geqslant 1$ and $1 \leqslant t \leqslant p - q - 1$.

Output: The forcing number of $H(p, q, t)$.

Step 0. Set $a := p-1$, $b := 1$, and $k := q+1$ (a is the minimal x-coordinate over all bottom-left vertices of trapeziums except P_1, b is the maximal x-coordinate over all the bottom-right vertices of the trapeziums except P_1, and k is the side length of the normal triangle).

Step 1. Set $s := \lfloor \frac{k}{q} \rfloor$ and $r := k - sq$; If $r = 0$, set $s := s - 1$, $r := q$.

Step 2. If $r = q$ and $0 \leqslant p-(s+1)t \pmod{p} \leqslant k$, obtain the characteristic triangle and output k, stop.

Step 3. If $r = 1$, set $a := \min\{a, p - st \pmod{p}\}$, $b := \max\{b+1, (p-st) \pmod{p} + 1\}$; else, set $b := b + 1$.

Step 4. If $a = k$ or $b = q$, obtain the characteristic triangle and output k, then stop.

Step 5. Set $k := k + 1$. Then go to step 1. □

A program of algorithm 5.1 in Microsoft Visual FoxPro 6.0 has been accomplished on micro computer.

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